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COMPACT GROUPS OF OPERATORS ON PROPORTIONAL QUOTIENTS OF l_1^n

BY

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ABSTRACT

It is proved that for arbitrary $m \in \mathbb{N}$ and for a sufficiently nontrivial compact group G of operators acting on a "typical" n-dimensional quotient X_n of l_1^m with $m = (1+\delta)n$, there is a constant $c = c(\delta)$ such that

$$\sup_{\|x\|=1} \int_G \|Tx\| dh_G(T) \geq c\sqrt{n}/\log n.$$

1. Introduction

We shall study some properties of quotients of l_1^m with the dimension proportional to m. While studing local theory of Banach spaces the following type of problems is considered: For a given class $\mathcal{G}$ of groups acting on an n-dimensional linear space and an n-dimensional Banach space X_n estimate the quantity

$$\mathrm{I}(X_n, \mathcal{G}) = \inf_{G \in \mathcal{G}} \sup_{\|x\|=1} \sup_{T \in G} \|Tx\|. \tag{1.1}$$

A typical example of such a situation is the case of groups connected with the symmetric or unconditional basis structure of X_n, [T]. In this paper we shall study the invariant

$$\mathrm{Av}(X_n, \mathcal{G}) = \inf_{G \in \mathcal{G}} \sup_{\|x\|=1} \int_G \|Tx\| dh_G(T), \tag{1.2}$$

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for "typical" proportional quotients of l_1^m of a fixed proportion and for some classes of compact groups of operators. Since l_1-norm for a normalized measure is smaller than l_∞-norm one has $\mathrm{Av}(X_n, \mathcal{G}) \le \mathrm{I}(X_n, \mathcal{G})$. We shall prove that if the class of groups $\mathcal{G}$ consists of sufficiently nontrivial groups only, then

$$\mathrm{Av}(X_n, \mathcal{G}) \ge \frac{c\sqrt{n}}{\log n}$$

for "most of" proportional quotients X_n of l_1^n. In particular, this holds true for the class of compact groups acting irreducibly on X_n or the class of groups connected with the unconditional basis structure. It is worth mentioning that our methods allow considering proportional quotients of arbitrary but fixed proportion while previously some essential restrictions on the proportion were required [Sz.1], [Sz.2], [M-T.1] and [M-T.2].

We shall deal with the real spaces only. However, after suitable modification the complex case follows by essentially the same argument.

2. Average invariants

We shall work mainly in $\mathbb{R}^n$. By $\|\cdot\|_p$ we shall denote the standard l_p-norm on $\mathbb{R}^n$ and we shall identify n-dimensional Banach spaces with $\mathbb{R}^n$ equipped with suitable norms, i.e. for an n-dimensional Banach space X_n we shall write $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$. The n-dimensional Hilbertian space H_n will be identified by $(\mathbb{R}^n, \|\cdot\|_2)$. For a Banach space X_n the set of extreme points of its unite ball B_{X_n} will be denoted by $\mathrm{Ex}(X_n)$ and the cardinality of $\mathrm{Ex}(X_n)$ will be denoted by $\mathrm{e}(X_n)$. We shall restrict our interest to the spaces with $\mathrm{e}(X_n) < \infty$. Note that if $\mathrm{e}(X_n)$ is finite then one can identify X_n with an appropriate quotient of $l_1^{\mathrm{e}(X_n)}$. If a Banach space $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$ is fixed, then for a linear subspace $E \subset \mathbb{R}^n$ we shall denote by X_n/E the corresponding quotient space of X_n. In particular, if B_{X_n} is the unit ball of X_n we shall identify X_n/E with the orthogonal complement $E^\perp$ of E equipped with the norm $\|\cdot\|_{P_{E^\perp}(B_{X_n})}$ induced on $E^\perp$ by $P_{E^\perp}(B_{X_n})$, where $P_{E^\perp}$ stands for the orthogonal projection onto $E^\perp$. The quotient map from X_n onto X_n/E will be denoted by q_E.

$\mathcal{G}_n$ will stand for the set of all compact groups of operators acting on $\mathbb{R}^n$. For $G \in \mathcal{G}_n$ the normalized Haar measure on G will be denoted by h_G. We shall say that a family $\mathcal{G} \subset \mathcal{G}_n$ is isomorphically invariant if for every compact group of operators $G \in \mathcal{G}$ and for every isomorphism $T\colon \mathbb{R}^n \to \mathbb{R}^n$ the group TGT^{-1} also belongs to $\mathcal{G}$. The class of all isomorphically invariant subsets of $\mathcal{G}_n$ will be denoted by $\mathcal{G}_n^0$.

For an n-dimensional Banach space $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$ and for a family of isomorphically invariant groups $\mathcal{G} \in \mathcal{G}_n^0$ we shall consider the quantity

$$\mathrm{Av}(X_n, \mathcal{G}) = \inf_{G\in\mathcal{G}} \sup_{\|x\|_{X_n}=1} \int_G \|Ux\|_{X_n} dh_G(U). \tag{2.1}$$

We shall need the following obvious properties of $\mathrm{Av}(X_n, \mathcal{G})$.

PROPOSITION 2.1: *For every isomorphically invariant family of groups $\mathcal{G}$ one has*

(i) $\mathrm{Av}(X_n, \mathcal{G}) \geq 1$ *for every Banach space* $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$;

(ii) $\mathrm{Av}(H_n, \mathcal{G}) = 1$;

(iii) *for every pair of Banach spaces* $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$ *and* $Y_n = (\mathbb{R}^n, \|\cdot\|_{Y_n})$

$$\mathrm{Av}(X_n, \mathcal{G}) \leq \mathrm{d}(X_n, Y_n)\, \mathrm{Av}(Y_n, \mathcal{G}),$$

where $\mathrm{d}(X_n, Y_n)$ *stands for the Banach Mazur distance between* X_n *and* Y_n;

(iv) *in particular*

$$\mathrm{Av}(X_n, \mathcal{G}) \leq \mathrm{d}(X_n, H_n).$$

Recall, that an operator $T \in L(\mathbb{R}^n)$ is said to be (k,α)-mixing (or briefly $T \in \mathrm{Mix}_n(k,\alpha)$), for some $k \geq 0$ and $\alpha > 0$, iff there exists a linear subspace $E \subset \mathbb{R}^n$ with $\dim(E) \geq k$ such that

$$\|P_{E^\perp} Tx\|_2 \geq \alpha \|x\|_2 \quad \text{for every } x \in E,$$

where $P_{E^\perp}$ stands for the orthogonal projection onto $E^\perp$. Following the definition of mixing operators, we shall say that a compact group of operators $G \in \mathcal{G}_n$ is (k,α,ρ)-mixing for some $k \geq 1$, $\alpha > 0$ and $\rho \in (0,1]$ (or briefly $G \in \mathrm{Mix}_n(k,\alpha,\rho)$) iff

$$h_G(\{T \in G | T \in \mathrm{Mix}_n(k,\alpha)\}) \geq \rho. \tag{2.2}$$

The following Theorem is a generalization of Theorem 1.4 in [Sz.2].

THEOREM 2.2: *For every $\delta, \alpha > 0$ and every $\kappa \in (0, 1/2]$ there is a constant $c_1 = c_1(\kappa,\delta)$ such that for every $n > 2$ there exists a Banach space $X_n = (\mathbb{R}^n, |\cdot|_{X_n})$ satisfying the properties*

(i) $\mathrm{e}(X_n) = (1+\delta)n$,

(ii) $\frac{1}{2}\|x\|_2 \leq |x|_{X_n} \leq \|x\|_1$ *for every* $x \in \mathbb{R}^n$,

(iii) *for every linear subspace E of $\mathbb{R}^n$ with $\dim E \leq \kappa n/32\log n$ the following estimate holds:*

$$\|q_E T\colon X_n \to X_n/E\| \geq c_1 \alpha\sqrt{n},$$

for every $T \in \mathrm{Mix}_n(\kappa n, \alpha)$.

The proof of this theorem is postponed to the last two sections. In fact, we shall prove that "most of" n-dimensional quotients of l_1^m with $m = (1+\delta)n$ satisfy the requirements of the theorem.

Remark: Note that in the language of "s-numbers" (iii) in the theorem above exactly means that the $(\kappa n/32\log n)$-th Kolmogorov number is at least $c_1\alpha\sqrt{n}$ for every operator $T \in \mathrm{Mix}_n(\kappa n, \alpha)$.

LEMMA 2.3: *Let $X_n = (\mathbb{R}^n, |\cdot|_{X_n})$ be a Banach space satisfying the thesis of Theorem 2.2 for some $\delta, \alpha > 0$ and $\kappa \in (0, 1/2]$. Then*

$$\mathrm{Card}\{x \in \mathrm{Ex}(X_n) | |Tx|_{X_n} \geq c_1\alpha\sqrt{n}\} > \frac{\kappa n}{32\log n},$$

for every $T \in \mathrm{Mix}_n(\kappa n, \alpha)$.

Proof: Fix arbitrary $T \in \mathrm{Mix}_n(\kappa n, \alpha)$ and set

$$E = \mathrm{lin}\{x \in \mathrm{Ex}(X_n) | |Tx|_n \geq c_1\alpha\sqrt{n}\}.$$

Clearly,

$$\|q_E T\colon X_n \to X_n/E\| < c_1\alpha\sqrt{n}$$

which, by Theorem 2.2 (iii), implies that $\dim E > \kappa n/32\log n$. ∎

The next result is a formal consequence of Theorem 2.2 and the previous lemma.

THEOREM 2.4: *For every $\delta, \alpha > 0, p \in (0,1), \kappa \in (0, 1/2]$ and every $n \geq 2$ there is a Banach space $X_n = (\mathbb{R}^n, |\cdot|_{X_n})$ satisfying the properties*

(i) $\mathrm{e}(X_n) = (1+\delta)n$;

(ii) $\frac{1}{2}\|x\|_2 \leq |x|_{X_n} \leq \|x\|_1$ *for every $x \in \mathbb{R}^n$;*

(iii) *for every group $G \in \mathrm{Mix}_n(\kappa n, \alpha, p)$ one has*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x\in \mathrm{Ex}(X_n)} \int_G |Tx|_{X_n} dh_G(T) \geq \frac{c_1\kappa\alpha p\sqrt{n}}{32(1+\delta)\log n},$$

where c_1 is the constant from Theorem 2.2.

Proof: Fix $\delta, \alpha > 0, p \in (0,1)$ and $\kappa \in (0, 1/2]$. Let X_n be a Banach space satisfying the conditions of Theorem 2.2. It remains to prove (iii). For every group $G \in \mathrm{Mix}_n(\kappa n, \alpha, p)$ we have

$$\begin{aligned}\frac{1}{\mathrm{e}(X_n)} \sum_{x\in \mathrm{Ex}(X_n)} \int_G |Tx|_{X_n} dh_G(T) &= \int_G \frac{1}{(1+\delta)n} \sum_{x\in \mathrm{Ex}(X_n)} |Tx|_{X_n} dh_G(T) \\ &\geq \int_{\mathcal{A}} \frac{1}{(1+\delta)n} \sum_{x\in \mathrm{Ex}(X_n)} |Tx|_{X_n} dh_G(T),\end{aligned} \tag{2.3}$$

where $\mathcal{A} = \{T \in G | T \in \mathrm{Mix}_n(\kappa n, \alpha, p)\}$. Since $G \in \mathrm{Mix}_n(\kappa n, \alpha, p)$ yields that $h_G(\mathcal{A}) \geq p$, by the previous lemma we infer that

$$\int_{\mathcal{A}} \frac{1}{(1+\delta)n} \sum_{x \in \mathrm{Ex}(X_n} |Tx|_{X_n} dh_G(T) \geq \int_{\mathcal{A}} \frac{\kappa}{32(1+\delta)\log n} c_1 \alpha \sqrt{n} dh_G(T)$$
$$\geq \frac{c_1 \kappa \alpha p \sqrt{n}}{32(1+\delta)\log n},$$

which, combined with (2.3), proves the required estimate. ∎

Remark: Note that Theorem 2.4 implies that if X_n is a Banach space satisfying the conditions of Theorem 2.2, then for every isomorphically invariant family of groups $\mathcal{G} \in \mathcal{G}_n^0$ with the property that for every $G \in \mathcal{G}$ one has $G \in \mathrm{Mix}_n(\kappa n, \alpha, p)$ for some fixed $\kappa, \alpha > 0$ and $p \in (0,1]$ the following lower estimate for $\mathrm{Av}(X_n, \mathcal{G})$ holds:

$$\mathrm{Av}(X_n, \mathcal{G}) \geq \frac{c_1 \kappa \alpha p \sqrt{n}}{32(1+\delta)\log n}.$$

On the other hand, by Proposition 2.1 (iv) and John's Theorem one has

$$\mathrm{Av}(X_n, \mathcal{G}) \leq \sqrt{n}.$$

Thus, for fixed κ, α and δ, the lower and upper estimates for $\mathrm{Av}(X_n, \mathcal{G})$ differ by a logarithmic factor. This leads to the following problem

PROBLEM 1: Does there exist a positive constant $c = c(\kappa, \alpha, p)$ with the property that for every positive integer n there is an m-dimensional Banach space Y_m with $m \geq n$ such that for every $\mathcal{G} \in \mathcal{G}_m^0$ with $G \in \mathrm{Mix}_n(\kappa m, \alpha, p)$ for every $G \in \mathcal{G}$ one has

$$\mathrm{Av}(Y_m, \mathcal{G}) \geq c\sqrt{m}?$$

3. Mixing properties of groups of operators

Our first example of a class of groups satisfying some mixing condition of the form (2.2) will be the class of groups connected with the notion of symmetry constants of finite-dimensional Banach spaces, [GG], [Ma.1]. Namely, recall that a compact group of operators $G \in \mathcal{G}_n$ is said to act irreducibly on $\mathbb{R}^n$ iff it does not admit a nontrivial invariant subspace. The class of all such groups of operators acting on $\mathbb{R}^n$ will be denoted by $\mathcal{G}_n^s$. Clearly, $\mathcal{G}_n^s \subset \mathcal{G}_n^0$. We have the following

THEOREM 3.1: *For every $G \in \mathcal{G}_n^s$ with $n \geq 3$ we have*

$$G \in \mathrm{Mix}_n\left(\frac{n}{20}, \frac{1}{4}, \frac{1}{5}\right).$$

Proof: Fix arbitrary $G \in \mathcal{G}_n^s$ with $n \geq 3$ and for every $x, y \in \mathbb{R}^n$ let

$$\langle x, y\rangle_G = \int_G \langle Ux, Uy\rangle_2 dh_G(U),$$

where $\langle \cdot, \cdot \rangle_2$ denotes the standard scalar product on $\mathbb{R}^n$. Set

$$\|x\|_G = \langle x, x\rangle_G^{1/2}.$$

Clearly, $\|\cdot\|_G$ is a Hilbertian norm on $\mathbb{R}^n$ and the group G consists of isometries of $(\mathbb{R}^n, \|\cdot\|_G)$. Hence the set $\mathcal{E} = \{x \in \mathbb{R}^n | \|x\|_G \leq 1\}$ is an ellipsoid. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ be the semiaxes of $\mathcal{E}$ arranged in nonincreasing order, i.e.

$$\|x_1\|_2 \geq \|x_2\|_2 \geq \cdots \geq \|x_n\|_2.$$

Set $E_1 = \operatorname{lin}\{x_j | j \leq n/2\}$ and $E_2 = \operatorname{lin}\{x_j | j > n/2\}$. Observe that the orthogonal projection P_{E_1} onto E_1 in $(\mathbb{R}^n, \|\cdot\|_G)$ is an orthogonal projection in $(\mathbb{R}^n, \|\cdot\|_2)$ as well. Consider the operator

$$T = \int_G U^{-1} P_{E_1} U dh_G(U).$$

In the sequel, in order to simplify notations we shall assume that n is an even number. One can easily verify that $TU = UT$ for every $U \in G$ and that $\operatorname{tr} T = n/2$. Therefore, cf. e.g., [Ma.2], Section 4, 1^0 (iii), one has

$$\langle Tx, x\rangle_G = \frac{\operatorname{tr} T}{n} \|x\|_G^2 = \frac{1}{2}\|x\|_G^2,$$

for every $x \in \mathbb{R}^n$. Thus, for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\frac{1}{2}\|x\|_G^2 = \langle Tx, x\rangle_G &= \int_G \langle U^{-1} P_{E_1} Ux, x\rangle_G dh_G(U) \\ = \int_G \langle P_{E_1} Ux, P_{E_1} Ux\rangle_G dh_G(U) &= \int_G \|P_{E_1} Ux\|_G^2 dh_G(U).\end{aligned}$$

Set $S_{E_2} = \{x \in E_2 | \|x\|_G = 1\}$ and note that S_{E_2} is the unit sphere in the $n/2$-dimensional Hilbertian space $(E_2, \|\cdot\|_G)$. Let μ_{E_2} be the normalized Haar measure on S_{E_2} with respect to the Hilbertian structure induced on S_{E_2} by $\|\cdot\|_G$. By the Fubbini Theorem we have

$$\begin{aligned}\frac{1}{2} &= \int_{S_{E_2}} \int_G \|P_{E_1} Ux\|_G^2 dh_G(U) d\mu_{E_1}(x) \\ &= \int_G \int_{S_{E_1}} \|P_{E_1} Ux\|_G^2 d\mu_{E_1}(x) dh_G(U) \\ &= \frac{2}{n} \int_G \|P_{E_1} U | E_2\|_{G,HS}^2 dh_G(U),\end{aligned}$$

where $\|\cdot\|_{G.HS}$ denotes the Hilbert-Schmidt norm of operators acting from $(E_2, \|\cdot\|_G)$ into $(\mathbb{R}^n, \|\cdot\|_G)$. Hence

$$\int_G \|P_{E_1}U|E_2\|^2_{G,HS}dh_G(U) = n/4. \tag{3.1}$$

On the other hand, since $P_{E_1}U|E_2$, for every $U \in G$, is a contraction with respect to the norm $\|\cdot\|_G$, we infer that

$$\|P_{E_1}U|E_2\|^2_{G,HS} \leq n/2. \tag{3.2}$$

Set $\mathcal{A} = \{U \in G | \|P_{E_1}U|E_2\|^2_{G,HS} \geq n/8\}$. (3.1) and (3.2) yield $h_G(\mathcal{A}) > 1/5$. To complete the proof of the theorem it suffices to show that every $U \in \mathcal{A}$ is $(n/20, 1/4)$-mixing. To this end, fix arbitrary $U \in \mathcal{A}$ and set $T_U = P_{E_1}U|E_2$. Write T_U in polar decomposition form with respect to the $\|\cdot\|_G$ norm

$$T_U x = \sum_{i=1}^{n/2} s_i \langle x, v_{2,i}\rangle_G v_{1,i},$$

where $s_1 \geq s_2 \geq \cdots \geq s_{n/2} \geq 0$ while $\{v_{1,i}\}$ and $\{v_{2,i}\}$ are orthonormal systems with respect to $\|\cdot\|_G$ norm in E_1 and E_2, respectively. Clearly $\|T_U\|^2_{G,HS} = \sum_{i=1}^{n/2} s_i^2 \geq n/8$. Since T_U is a contraction with respect to $\|\cdot\|_2$ norm we infer that $s_1 \leq 1$. Therefore, at least $n/20$ of s_i's are greater than or equal to $1/4$. Let $F = \text{lin}\{v_{2,1}, v_{2,2}, \dots, v_{2,n/20}\}$. For every $x \in F$, by the definition of spaces E_1 and E_2, we have

$$\begin{aligned}\|P_{F^\perp}Ux\|_2 &\geq \|P_{E_1}Ux\|_2 \geq \|x_{n/2}\|_2\|P_{E_1}Ux\|_G = \|x_{n/2}\|_2\|T_Ux\|_G \\ &\geq \tfrac{1}{4}\|x_{n/2}\|_2\|x\|_G \geq \tfrac{1}{4}\|x_{n/2}\|_2\|x_{n/2+1}\|_2^{-1}\|x\|_2 \geq \tfrac{1}{4}\|x\|_2,\end{aligned}$$

which means that $U \in \text{Mix}_n(n/20, 1/4)$ and completes the proof of the theorem. ∎

As an immediate consequence we obtain

THEOREM 3.2: *For every $\delta > 0$ there exist a constant $c = c(\delta)$ with the property that for every $n \geq 3$ there exists an n-dimensional quotient X_n of $l_1^{(1+\delta)n}$ such that*

$$\text{Av}(X_n, \mathcal{G}_n^s) \geq c\frac{\sqrt{n}}{\log n}.$$

Proof: Let X_n be a space satisfying Theorem 2.4. Fix arbitrary group $G \in \mathcal{G}_n^s$. By Theorem 3.1 we have $G \in \mathrm{Mix}_n(n/20, 1/4, 1/5)$. Thus, by Theorem 2.4

$$\sup_{\|x\|=1} \int_G |Tx|_{X_n} dh_G(T) \geq \frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_G |Tx|_{X_n} dh_G(T)$$
$$\geq \frac{c_1(1/20, \delta)\sqrt{n}}{12800(1+\delta)\log n}.$$

Hence

$$\mathrm{Av}(X_n, \mathcal{G}_n^s) \geq c(\delta)\frac{\sqrt{n}}{\log n},$$

where $c = c_1(1/20, \delta)/12800(1+\delta)$. ∎

The author does not know the answer to the following

PROBLEM 2: Does there exist $c = c(\delta) > 0$ such that for every $\delta > 0$ and every positive integer n there exists an n-dimensional quotient X_n of $l_1^{(1+\delta)n}$ with the property

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_G |Tx|_{X_n} dh_G \geq c\sqrt{n},$$

for every group $G \in \mathcal{G}_n^s$?

PROBLEM 3: Does there exist $c > 0$ such that there are finite-dimensional Banach spaces Y_m with arbitrary large dimension m with the property

$$\mathrm{Av}(Y_m, \mathcal{G}_m^s) \geq c\sqrt{m}?$$

Another example of a class of groups satisfying some mixing conditions are the groups connected with the unconditional basis structure. Namely, let

$$C_n = \{\epsilon = (\epsilon_1, \epsilon_2, \ldots \epsilon_n) | \epsilon_i \in \{-1, 1\} \text{ for } i = 1, 2, \ldots, n\}$$

be the n-th Cantor group. For an arbitrary fixed basis $\{x_i\}_{i=1}^n$ in $\mathbb{R}^n$ with dual functionals $\{x_i^*\}_{i=1}^n$ and for each $\epsilon \in C_n$ consider operator

$$T_{\epsilon,\{x_i\}}(x) = \sum_{i=1}^n \epsilon_i x_i^*(x) x_i.$$

Clearly, operators $\{T_{\epsilon,\{x_i\}}\}_{\epsilon \in C_n}$ form a compact group, say $G_{\{x_i\}}$, and the map $\phi: \epsilon \to T_{\epsilon,\{x_i\}}$ is a group isomorphism from C_n^* onto $G_{\{x_i\}}$. The class of all compact groups of that form will be denoted by $\mathcal{G}_n^{ub}$. Obviously, $\mathcal{G}_n^{ub}$ is isomorphically invariant.

THEOREM 3.3: *There is a constant $p_0 > 0$ such that for every $n \geq 2$*

$$G \in \mathrm{Mix}_n\left(\frac{n}{4}, 1, p_0\right)$$

for every $G \in \mathcal{G}_n^{ub}$.

Proof: Fix arbitrary $G \in \mathcal{G}_n^{ub}$. Let $G = G_{\{x_i\}}$ for some basis $\{x_i\}_{i=1}^n$ in $\mathbb{R}^n$. Set

$$\mathcal{A} = \{T_{\epsilon,\{x_i\}} \in G \,|\, |\mathrm{tr}(T_{\epsilon,\{x_i\}})| \leq n/2\} \tag{3.3}$$

and note that for $T_{\epsilon,\{x_i\}} \in G$ one has

$$T_{\epsilon,\{x_i\}} = \mathrm{Id}_{\mathbb{R}^n} - 2P_{\epsilon,\{x_i\}}, \tag{3.4}$$

where $P_{\epsilon,\{x_i\}}$ is a projection in $\mathbb{R}^n$ onto $\mathrm{lin}\{x_i | i \text{ such that } \epsilon_i = -1\}$ with kernel equal to $\mathrm{lin}\{x_i | i \text{ such that } \epsilon_i = 1\}$. One can easily show that every rank k projection P_k, with $k \leq n/2$, in an n-dimensional Hilbert space belongs to the mixing class $\mathrm{Mix}_n(k, 1/2)$. Indeed, trivially this is true for every 1-dimensional projection in $(\mathbb{R}^2, \|\cdot\|_2)$ and the general case follows from the observation that every rank k projection P_k, $k \leq n/2$, in $(\mathbb{R}^2, \|\cdot\|_2)$ is a direct orthogonal sum of k rank 1 projections in $(\mathbb{R}^n, \|\cdot\|_2)$. Since every projection with rank greater than $n/2$ in $\mathbb{R}^n$ is of the form $\mathrm{Id}_{\mathbb{R}^n} - P$, where P is a projection with rank less than or equal to $n/2$, and since adding a multiple of the identity operator does not change the mixing class of an operator, we infer that every projection P in an n-dimensional Hilbert space belongs to the mixing class $\mathrm{Mix}_n(k, 1/2)$, where $k = \min\{\mathrm{rank}\, P, n - \mathrm{rank}\, P\}$. Therefore, by (3.4), we infer that for every $T_{\epsilon,\{x_i\}} \in G$

$$T_{\epsilon,\{x_i\}} \in \mathrm{Mix}_n(k, 1),$$

where $k = (n - |\mathrm{tr}(T_{\epsilon,\{x_i\}})|)/2$. Thus $T_{\epsilon,\{x_i\}} \in \mathrm{Mix}_n(n/4, 1)$ for every $T_{\epsilon,\{x_i\}} \in \mathcal{A}$. To complete the proof it is enough to observe that $\mathrm{tr}(T_{\epsilon,\{x_i\}}) = \sum_{i=1}^n \epsilon_i$ and that, by the Law of Large Numbers, there exists $p_0 > 0$ such that

$$h_{G_{\{x_i\}}}(\mathcal{A}) = h_{C_n}\left(\left\{\epsilon \in C_n \middle| \left|\sum_{i=1}^n \epsilon_i\right| \leq \frac{n}{2}\right\}\right) \geq p_0,$$

for every $n \geq 2$. ∎

Similarly, as in the case of Theorem 3.2 the theorem above formally yields

THEOREM 3.4: *For every $\delta > 0$ there exists a constant $c = c(\delta)$ with the property that for every $n \geq 2$ there exists an n-dimensional quotient X_n of $l_1^{(1+\delta)n}$ such that*

$$\mathrm{Av}(X_n, \mathcal{G}_n^{ub}) \geq c\frac{\sqrt{n}}{\log n}.$$

Remark: Theorem 3.4 is not optimal. Namely, in [BKPS], for an n-dimensional Banach space Y_n, the quantity $\mathrm{Av}(Y_n, \mathcal{G}_n^{ub})$ is denoted by $ruc(Y_n)$ and K. Ball, in [B], proved that there are proportional quotients Y_n of l_1^m with $\mathrm{Av}(Y_n, \mathcal{G}_n^{ub})$ of the order of $\sqrt{n}$.

Nevertheless, the following problem remains open:

PROBLEM 4: Does there exist $c = c(\delta) > 0$ such that for every $\delta > 0$ and every positive integer n there exists an n-dimensional quotient X_n of $l_1^{(1+\delta)n}$ with the property

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_G |Tx|_{X_n} dh_G \geq c\sqrt{n},$$

for every group $G \in \mathcal{G}_n^{ub}$?

4. The main technical theorem

We shall work with a fixed probability space $(\mathbf{P}, \Omega)$. By a standard Gaussian vector in an n-dimensional Euclidean space H_n we shall mean an H_n-valued random variable with the density

$$(n/2\pi)^{n/2} \exp(-n\|x\|_2^2/2)$$

with respect to the standard Lebesgue measure on H_n. Some well-known basic properties of standard Gaussian vectors which will be used in the sequel are described in the following

PROPOSITION 4.1: *Fix $n \in \mathbb{N}$ and let g be a standard Gaussian vector in H_n. Then*

(i) *for every pair E, F of orthogonal subspaces in H_n the random vectors $P_E g$ and $P_F g$ are independent,*

(ii) *for every k-dimensional subspace E of H_n the random vector*

$$\sqrt{\frac{n}{k}} P_E g$$

is a standard Gaussian vector in E,

(iii) *there exists an absolute constant $0 < c_0 < 1$ such that*

$$\mathbf{P}(\{\omega \in \Omega \colon 1/2 \leq \|g(\omega)\|_2 \leq 2\}) > 1 - c_0^n.$$

In the sequel, for a given, fixed $\delta > 0$ and $n \in \mathbb{N}$ we shall consider δn independent standard Gaussian vectors $g_1, g_2, \ldots, g_{\delta n}$ in $\mathbb{R}^n$. For every $\omega \in \Omega$ we define

$$B_{X(\omega)} = \mathrm{absconv}\{e_1, e_2, \ldots, e_n, g_1, g_2, \ldots, g_{\delta n}\},$$

where $e_1, e_2, \ldots, e_n$ is a standard unite vector basis in $\mathbb{R}^n$. The norm on $\mathbb{R}^n$ corresponding to $B_{X(\omega)}$ will be denoted by $\|\cdot\|_{X(\omega)}$ and we shall write $X(\omega) = (\mathbb{R}^n, \|\cdot\|_{X(\omega)})$, i.e. $X(\omega)$ is the Banach space with the unit ball $B_{X(\omega)}$. Obviously, $X(\omega)$ is isometrically isomorphic to a quotient of $l_1^{(1+\delta)n}$. Note that

$$\frac{1}{\sqrt{n}} B_2^n \subset B_1^n \subset B_{X(\omega)}, \tag{4.1}$$

for every $\omega \in \Omega$.

We shall prove the following probabilistic version of Theorem 2.2:

THEOREM 4.2: *For every $\alpha, \delta > 0$ and $0 < \kappa < 1/2$ there exist constants $c_1 = c_1(\kappa, \delta), c_2 = c_2(\kappa, \delta) > 0$ such that if $m = \kappa n / 32 \log n$ and $\tilde{\Omega}(\kappa, \delta)$ denotes the set*

$$\tilde{\Omega}(\kappa, \delta) = \{\omega \in \Omega | \|q_F T \colon X(\omega) \to X(\omega)/F\| \geq c_1 \alpha n^{1/2}$$
$$\text{for every } T \in Mix_n(\kappa, \alpha) \text{ and every } F \subset \mathbb{R}^n \text{ with } \dim F \leq m\},$$

then $\mathbf{P}(\tilde{\Omega}(\kappa, \delta)) \geq 1 - c_2^n$.

Before we are able to proceed with the proof of Theorem 4.2 we shall need some more notation. First, observe that it is enough to prove Theorem 4.2 for $\alpha = 1$. For given $\delta > 0$ and $0 < \kappa < 1/2$ we shall write $m = \kappa n / 32 \log n$ and $\varepsilon = \min\{\kappa n / 32, \delta\}$. Due to the probabilistic nature of our argument, without any loss of generality we may and shall assume that

$$(\mathbf{P}, \Omega) = (\mathbf{P}_1, \Omega_1) \times (\mathbf{P}_2, \Omega_2),$$

where $(\mathbf{P}_1, \Omega_1)$ and $(\mathbf{P}_2, \Omega_2)$ are probabilistic spaces. Consequently, we shall write $\omega = (\omega_1, \omega_2)$. Moreover, we shall assume that g_j's for $j = 1, 2, \ldots, \varepsilon n$ depend on ω_1 only, i.e. $g_j(\omega_1, \omega_2) = \tilde{g}_j(\omega_1)$, and similarly we shall assume that g_j's for $j = \varepsilon n + 1, \varepsilon + 2, \ldots, \delta n$ depend on ω_2 only, i.e. $g_j(\omega_1, \omega_2) = \tilde{g}_j(\omega_2)$. Set

$$\Omega_1' = \{\omega_1 \in \Omega_1 | \tfrac{1}{2} \leq \|\tilde{g}_j(\omega_1)\|_2 \leq 2 \text{ for every } j = 1, 2, \ldots, \varepsilon n\}$$

and

$$\Omega_2' = \{\omega_2 \in \Omega_2 | \tfrac{1}{2} \leq \|\tilde{g}_j(\omega_2)\|_2 \leq 2 \text{ for every } j = \varepsilon n + 1, \ldots, \delta n\}.$$

By Proposition 4.1 (iii), we have

$$\mathbf{P}_1(\Omega_1') \geq 1 - \varepsilon n c_0^n \tag{4.2}$$

and similarly

$$\mathbf{P}_2(\Omega_2') \geq 1 - (\delta - \varepsilon) n c_0^n. \tag{4.3}$$

We shall prove the following theorem which easily implies Theorem 4.2.

THEOREM 4.3: *For every $\omega_2 \in \Omega_2'$, $\delta > 0$ and $0 < \kappa < 1/2$ there exist constants $c_1 = c_1(\kappa,\delta), c_2 = c_2(\kappa,\delta) > 0$ such that if $\tilde{\Omega}_1(\omega_2,\kappa,\delta)$ denotes the set*

$$\tilde{\Omega}_1(\omega_2,\kappa,\delta) = \{\omega_1 \in \Omega_1' | \|q_F T\colon X(\omega_1,\omega_2) \to X(\omega_1,\omega_2)/F\| \geq c_1 n^{1/2}$$
$$\text{for every } T \in \mathrm{Mix}_n(\kappa,1) \text{ and every } F \subset \mathbb{R}^n \text{ with } \dim F \leq m\},$$

then $\mathbf{P}_1(\tilde{\Omega}_1(\omega_2,\kappa,\delta)) \geq 1 - c_2^n$.

For every $\omega_1 \in \Omega_1$ define $E_{\omega_1} = \mathrm{span}\{\tilde{g}_1(\omega_1), \tilde{g}_2(\omega_1), \ldots, \tilde{g}_{\varepsilon n}(\omega_1)\}$. Let $G_{m,n}$ be the Grassmann manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. For every $F \in G_{m,n}$ let $Q_{\omega_1,F}$ be the orthogonal projection in $\mathbb{R}^n$ with $\ker Q_{\omega_1,F} = F + E_{\omega_1}$.

PROPOSITION 4.4: *For every $\omega_2 \in \Omega_2'$, $0 < \beta < 1$ and for every operator $T \in \mathrm{Mix}_n(\kappa n/2, 1)$ define*

$$\mathcal{A}(\omega_2,T,\beta) = \{\omega_1 \in \Omega_1' | \|Q_{\omega_1,F} T\colon X(\omega_1,\omega_2) \to Q_{\omega_1,F}(X(\omega_1,\omega_2))\| \leq 2\beta n^{1/2}$$
$$\text{for some } F \in G_{m,n}\}.$$

Then there is a constant $C = C(\kappa,\delta)$ such that for every $\omega_2 \in \Omega_2'$ and every $T \in \mathrm{Mix}_n(\kappa n/2,1)$ one has

$$\mathbf{P}_1(\mathcal{A}(\omega_2,T,\beta)) < C^{n^2} \beta^{\varepsilon\kappa n^2/16}.$$

The proof of Proposition 4.4 is postponed to the last section.

For every $\omega_2 \in \Omega_2'$ we let

$$B_{Y(\omega_2)} = \mathrm{absconv}\{e_1, e_2, \ldots, e_n, \tilde{g}_{\varepsilon n+1}(\omega_1), \tilde{g}_{\varepsilon n+2}(\omega_1), \ldots, \tilde{g}_{\delta n}(\omega_1)\}$$

and denote by $\|\cdot\|_{Y(\omega_2)}$ the norm on $\mathbb{R}^n$ induced by $B_{Y(\omega_2)}$. We shall write $Y(\omega_2) = (\mathbb{R}^n, \|\cdot\|_{Y(\omega_2)})$. Since the ball $B_{Y(\omega_2)}$ is the absolute convex of $(1+\delta-\varepsilon)n$ points each of them of Euclidean length not exceeding 2, by a well known argument [G], we infer that there exists a constant $C_1 = C_1(\kappa,\delta)$ such that

$$\mathrm{vol}(B_{Y(\omega_2)}) < \left(\frac{C_1}{n}\right)^n. \tag{4.4}$$

The next lemma is a standard ingredient (cf. [G], [M-T.1], [Sz.2]).

LEMMA 4.5: *For every $\omega_2 \in \Omega_2$ and $A > 0$ let*

$$\mathcal{T}_{\omega_2,A} = \{T\colon \mathbb{R}^n \to \mathbb{R}^n | T \in \mathrm{Mix}_n(\kappa n/2,1) \text{ and } \|T\colon l_1^n \to Y(\omega_2)\| \leq A\}.$$

Then there exists a constant $C_2 = C_2(\kappa,\delta)$ such that $\mathcal{T}_{\omega_2,A}$ admits a $An^{-1/2}/4$-net $\mathcal{N}_{\omega_2,A}$ with respect to the l_2^n operator norm with cardinality

$$\mathrm{card}\,\mathcal{N}_{\omega_2,A} \leq C_2^{n^2}.$$

Proof: Identifying operators acting on $\mathbb{R}^n$ with $n \times n$-matrices and consequently with points in $\mathbb{R}^{n^2}$, by (4.4), we infer that for every $\omega_2 \in \Omega_2'$

$$\mathrm{vol}_{n^2}(\mathcal{T}_{\omega_2,A}) < \left(\frac{C_1 A}{n}\right)^{n^2}.$$

On the other hand, it is well known (cf. [G]) that

$$\mathrm{vol}_{n^2}(\{T\colon \mathbb{R}^n \to \mathbb{R}^n \,|\, \|T\|_2 \le 1\}) \le \left(\frac{C'}{\sqrt{n}}\right)^{n^2},$$

where C' is some numerical constant and the required estimate follows from standard arguments on the cardinality of minimal nets. ∎

Proof of Theorem 4.3: Assuming the validity of Proposition 4.4 we shall prove Theorem 4.3. Obviously, it is enough to prove the theorem for sufficiently large n. Fix arbitrary $\omega_2 \in \Omega_2'$, $0 < \kappa < 1/2$ and $\delta > 0$. Let C and C_2 be the constants from Proposition 4.4 and Lemma 4.5, respectively. Set $\beta = (2CC_2)^{-16/\varepsilon\kappa}$ and $A = 2\beta n^{1/2}$. Let $\mathcal{N}_{\omega_2,A}$ be the $An^{-1/2}/4$-net in $\mathcal{T}_{\omega_2,A}$ from Lemma 4.5. Set

$$\Omega_1''(\omega_2, A) = \Omega_1' \setminus \bigcup_{T\in\mathcal{N}_{\omega_2,A}} \mathcal{A}(\omega_2, T, \beta),$$

where for $T \in \mathcal{N}_{\omega_2,A}$ the sets $\mathcal{A}(\omega_2, T, \beta)$ are defined in Proposition 4.4. By (4.2), Lemma 4.5 and the choice of β we infer that there exists a constant $c = c(\kappa, \delta)$ such that

$$\mathbf{P}_1(\Omega_1''(\omega_2, A)) \ge (1 - \varepsilon n c_0^n) - (1/2)^{n^2} > 1 - c^n,$$

for sufficiently large n. To complete the proof of Theorem 4.3 with $c_1 = c_1(\kappa, \delta) = \beta$ it is enough to show that $\Omega_1''(\omega_2, A) \subset \tilde{\Omega}_1(\omega_2, \kappa, \delta)$. Hence it suffices to prove that for every $\omega_1 \in \Omega_1''(\omega_2, A)$, every $T \in \mathrm{Mix}_n(\kappa n, 1)$ and every $F \in G_{m,n}$ one has

$$\|Q_{\omega_1,F}T\colon X(\omega_1,\omega_2) \to Q_{\omega_1,F}(X(\omega_1,\omega_2))\| \ge \beta n^{1/2}. \tag{4.5}$$

Assume to the contrary that there exist $\omega_{1,0} \in \Omega_1''(\omega_2, A)$, $T_0 \in \mathrm{Mix}_n(\kappa n, 1)$ and $F_0 \in G_{m,n}$ such that

$$\|Q_{\omega_{1,0},F_0}T_0\colon X(\omega_{1,0},\omega_2) \to Q_{\omega_{1,0},F_0}(X(\omega_{1,0},\omega_2))\| < \beta n^{1/2}. \tag{4.6}$$

To simplify the notation we shall write Q for $Q_{\omega_{1,0},F_0}$. Clearly $Q(X(\omega_{1,0},\omega_2)) = Q(Y(\omega_2))$, By (4.1), we have

$$\|QT_0\colon l_1^n \to Q(Y(\omega_2))\| < \beta n^{1/2}.$$

By a standard lifting argument we obtain an operator T_1: $\mathbb{R}^n \to \mathbb{R}^n$ with the properties $QT_0 = QT_1$ and $\|T_1: l_1^n \to Y(\omega_2)\| < \beta n^{1/2}$. Observe that $\operatorname{rank}(T_0 - T_1) \le \ker Q \le \kappa n/8$ implies $T_1 \in \mathrm{Mix}_n(\kappa n/2, 1)$. Thus $T_1 \in \mathcal{T}_{\omega_2,A}$. Choose $T \in \mathcal{N}_{\omega_2,A}$ such that $\|T_1 - T\|_2 < \beta/2$ and note that $\omega_{1,0} \in \Omega_1''(\omega_2, A)$ yields

$$\|QT: X(\omega_{1,0}, \omega_2) \to Q(X(\omega_{1,0}, \omega_2))\| > 2\beta n^{1/2}. \tag{4.7}$$

On the other hand, for $(\omega_1, \omega_2) \in \Omega_1' \times \Omega_2'$ we have

$$n^{-1/2} B_2^n \subset B_{X(\omega_1,\omega_2)} \subset 2B_2^n.$$

Therefore

$$\|QS: X(\omega_{1,0}, \omega_2) \to Q(X(\omega_{1,0}, \omega_2))\| \le 2n^{1/2}\|S\|_2, \tag{4.8}$$

for every S: $\mathbb{R}^n \to \mathbb{R}^n$. Since $QT_0 = QT_1$, combining (4.7) and (4.8) we obtain

$$\begin{aligned}
&\|QT_0: X(\omega_{1,0}, \omega_2) \to Q(X(\omega_{1,0}, \omega_2))\| \\
\ge &\|QT: X(\omega_{1,0}, \omega_2) \to Q(X(\omega_{1,0}, \omega_2))\| \\
&- \|(QT_1 - QT): X(\omega_{1,0}, \omega_2) \to Q(X(\omega_{1,0}, \omega_2))\| \\
> &2\beta n^{1/2} - \beta n^{1/2} = \beta n^{1/2},
\end{aligned}$$

which contradicts (4.6) and proves (4.5), completing the proof of Theorem 4.3. ∎

5. Proof of Proposition 4.4

Throughout this section we shall deal with a fixed $\omega_2 \in \Omega_2'$ and a fixed operator $T \in \mathrm{Mix}_n(\kappa n/2, 1)$. Note that by the definition of the mixing class $\mathrm{Mix}_n(\kappa n, 1)$ there is a linear subspace $E = E(T) \subset \mathbb{R}^n$ with $\dim E = \kappa n/2$ such that $\|P_{E^\perp} Tx\|_2 \ge \|x\|_2$ for every $x \in E$. A standard argument on circular sections of an ellipsoid yields that in such a case there exists a linear subspace $E_0 = E_0(T) \subset E$ with $\dim E_0 = \kappa n/4$ and a $\lambda = \lambda(T) \ge 1$ such that $\|P_{E^\perp} Tx\|_2 = \lambda \|x\|_2$ for every $x \in E_0$.

For every $\omega_1 \in \Omega_1$ and every $F \in G_{m,n}$ let Q_{ω_1,F,E_0} be the orthogonal projection in $\mathbb{R}^n$ with

$$\ker Q_{\omega_1,F,E_0} = E_{\omega_1} + E + P_{E^\perp} T P_{E_0^\perp} + F.$$

For $j = 1, 2, \ldots, \varepsilon n$ define $g_j' = P_{E_0} g_j$ and $g_j'' = P_{E_0^\perp} g_j$. By Proposition 4.1 (i) we infer that the random vectors $\{g_1', g_2', \ldots, g_{\varepsilon n}', g_1'', g_2'', \ldots, g_{\varepsilon n}''\}$ are mutually independent. In the sequel we shall use the following simple observation.

Remark: For fixed ω_1, F and T the projection Q_{ω_1,F,E_0} is independent of g'_j's (i.e. depends on g''_j's only).

Indeed, $P_{E_0^\perp}(E_{\omega_1})$ depends on the distributions of g''_j's only and hence so does $P_{E^\perp}TP_{E_0^\perp}(E_{\omega_1})$. On the other hand, since $E_0 \subset E$

$$E_{\omega_1} + E = P_{E^\perp}(E_{\omega_1}) + E = P_{E^\perp}P_{E_0^\perp}(E_{\omega_1}) + E$$

and, by the same token, $E_{\omega_1} + E$ depends on the distributions of g''_j's only.

LEMMA 5.1: *For every $\beta > 0$, $T \in \mathrm{Mix}_n(\kappa n/2, 1)$, $F \in G_{m,n}$ and $\omega_2 \in \Omega'_2$ let*

$$\mathcal{A}(T, F, \omega_2) = \{\omega_1 \in \Omega'_1 | Q_{\omega_1,F,E_0}TP_{E_0}g_j \in 4\lambda\beta n^{1/2}Q_{\omega_1,F,E_0}B_{X(\omega_1,\omega_2)} \text{ for every } j = 1, 2, \ldots, \varepsilon n\}.$$

Then there exists a constant $c' = c'(\kappa, \varepsilon)$ such that

$$\mathbf{P}_1(\mathcal{A}(T, F, \omega_2)) \leq (c'\beta)^{\varepsilon\kappa n^2/8}.$$

Proof: In order to simplify the notation we shall write Q_{ω_1} for Q_{ω_1,F,E_0}. For every fixed $j = 1, 2, \ldots, \varepsilon n$ we have

$$\begin{aligned} &\{\omega_1 \in \Omega_1 | Q_{\omega_1}TP_{E_0}g_j \in 4\lambda\beta n^{1/2}Q_{\omega_1}B_{X(\omega_1,\omega_2)}\} \\ (5.1)\quad =&\{\omega_1 \in \Omega_1 | Q_{\omega_1}Tg'_j \in 4\lambda\beta n^{1/2}Q_{\omega_1}B_{X(\omega_1,\omega_2)}\} \\ =&\{\omega_1 \in \Omega_1 | \lambda^{-1}Q_{\omega_1}T\sqrt{n/\dim E_0}\,g'_j \in 8\kappa^{-1/2}\beta n^{1/2}Q_{\omega_1}B_{X(\omega_1,\omega_2)}\}. \end{aligned}$$

Set $S = \lambda^{-1}Q_{\omega_1}T|E_0\colon E_0 \to \mathbb{R}^n$ and note that

(i) by Proposition 4.1 (ii), $\sqrt{n/\dim E_0}\,g'_j$ is a standard gaussian variable in E_0,

(ii) S has at least k s-numbers equal to 1 with $k \geq \kappa n/4 - m - 2\varepsilon n \geq \kappa n/8$,

(iii) the set $Q_{\omega_1}B_{X(\omega_1,\omega_2)}$ is the absolute convex hull of vectors $Q_{\omega_1}e_i$ for $i = 1, 2, \ldots, n$ and $Q_{\omega_1}g_i$ for $i = \varepsilon n + 1, \varepsilon + 2, \ldots, \delta n$, each of them with length not greater 2. In particular, $Q_{\omega_1}B_{X(\omega_1,\omega_2)}$ is independent of the distribution of g'_j.

By a simple modification of Claim 6.3 in [Sz.1] we infer that there exists a constant $c' = c'(\kappa, \delta)$ such that

$$(5.2)\quad \begin{aligned} &\mathbf{P}_1(\{\omega_1 \in \Omega_1 | \lambda^{-1}Q_{\omega_1}T\sqrt{n/\dim E_0}\,g'_j \in 8\kappa^{-1/2}\beta n^{1/2}Q_{\omega_1}B_{X(\omega_1,\omega_2)}\}) \\ &\qquad \leq (c'\beta)^{\kappa n/8}. \end{aligned}$$

Since g'_j's are independent, combining (5.1) and (5.2) we have

$$\begin{aligned} \mathbf{P}_1(\mathcal{A}(T, F, \omega_2)) &\leq \mathbf{P}_1(\{\omega_1 \in \Omega_1 | Q_{\omega_1,F,E_0}TP_{E_0}g_j \in 4\lambda\beta n^{1/2}Q_{\omega_1,F,E_0}B_{X(\omega_1,\omega_2)} \\ &\qquad \text{for every } j = 1, 2, \ldots, \varepsilon n\}) \\ &\leq (c'\beta)^{\varepsilon\kappa n^2/8}, \end{aligned}$$

which completes the proof of the lemma. ∎

The next lemma is a well known fact on nets on Grassmann manifolds due to Szarek, Lemma 7.3 in [Sz.1].

LEMMA 5.2: *There exists an absolute constant $C > 1$ such that for every $0 < \eta < 1$ and every $k < n$ the set $G_{k,n}$ admits an η-net $\mathcal{N}$ with the cardinality $\operatorname{Card}\mathcal{N} \le C^{n^2}\eta^{-kn}$ (with respect to the metric $\rho(F_1, F_2) = \|P_{F_1^\perp} - P_{F_2^\perp}\|_2$).*

Proof of Proposition 4.4: Fix any η-net $\mathcal{N}$ in $G_{m,n}$ with $\eta = \beta n^{-1/2}/4$. To prove the proposition it suffices to show that for arbitrary $T \in \mathrm{Mix}_n(\kappa n/2, 1)$ and every fixed $\omega_2 \in \Omega_2'$ one has

$$\mathcal{A}(T, \omega_2) \subset \bigcup_{F \in \mathcal{N}} \mathcal{A}(T, F, \omega_2). \tag{5.3}$$

Indeed, Lemmas 5.1, 5.2 and formula (5.3) yield

$$\mathbf{P}_1(\mathcal{A}(T, \omega_2)) \le C^{n^2}(\beta n^{-1/2}/4)^{-mn}(c'\beta)^{\varepsilon\kappa n^2/8}.$$

The last inequality easily implies the existence of a constant $C = C(\kappa, \delta)$ such that the estimate required in Proposition 4.4 holds for every n satisfying $\log n > \varepsilon^{-1}$.

To complete the proof of the proposition it remains to prove (5.3). To this end fix $\tilde{\omega}_1 \in \mathcal{A}(T, \omega_2)$ and choose $F_0 \in G_{m,n}$ such that

$$\|Q_{\tilde{\omega}_1,\omega_2,F_0}T\colon X(\tilde{\omega}_1, \omega_2) \to Q_{\tilde{\omega}_1,\omega_2,F_0}(X(\tilde{\omega}_1, \omega_2))\| \le 2\beta n^{1/2}. \tag{5.4}$$

Pick $F_1 \in \mathcal{N}$ such that $\rho(F_0, F_1) < \eta$ and, to simplify notation, write Q_i, $i = 0, 1$, for $Q_{\tilde{\omega}_1,\omega_2,F_i}$. Clearly, $\|Q_0 - Q_1\|_2 < \eta$. Hence for every $x \in \mathbb{R}^n$ we have $\|Q_1Q_0x - Q_1x\|_2 \le \eta\|x\|_2$. On the other hand, $\tilde{\omega}_1 \in \Omega_1'$ and $\omega_2 \in \Omega_2'$ yields $n^{-1/2}B_2^n \subset B_{X(\tilde{\omega}_1,\omega_2)} \subset 2B_2^n$. Thus

$$Q_1Q_0(B_{X(\tilde{\omega}_1,\omega_2)}) \subset Q_1(B_{X(\tilde{\omega}_1,\omega_2)}) + 2\eta Q_1(B_2^n). \tag{5.5}$$

For $j = 1, 2, \ldots, \varepsilon n$ let $x_j = TP_{E_0}g_j(\tilde{\omega}_1)$. Then $\|x_j\|_2 \le 2\lambda$. Observe that $P_{E^\perp}Q_0x_j$ and $P_EQ_0x_j$ belong to $\ker Q_0$ for $j = 1, 2, \ldots, \varepsilon n$ and therefore

$$Q_0x_j = Q_0Tg_j \quad \text{for } j = 1, 2, \ldots, \varepsilon n. \tag{5.6}$$

Combining (5.4), (5.5) and (5.6), we obtain that

$$\begin{aligned} Q_1x_j &= Q_1((Q_1 - Q_0)x_j - Q_0Tg_j(\tilde{\omega}_1)) \\ &\in 2\lambda\eta Q_1(B_2^n) + 2\beta n^{1/2}Q_1Q_0(B_{X(\tilde{\omega}_1,\omega_2)}) \\ &\subset 4\beta\lambda n^{1/2}Q_1(B_{X(\tilde{\omega}_1,\omega_2)}), \end{aligned}$$

for every $j = 1, 2, \ldots, \varepsilon n$, which means that $\tilde{\omega}_1 \in \mathcal{A}(T, F, \omega_2)$ and completes the proof of (5.3). ∎

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